Making sense of . . .

(Common) Shock models in Exam MLC

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Foreword

Based on released SoA MLC/M/3 exams, it appears that a question involving shock models appears only every third exam or so, while questions on independent joint lives appear five or six times as often. Candidates sometimes decide, therefore, to skip shock models—especially since their descriptions in some text materials seem very complicated—but to master joint statuses with independent lives. But in fact the shock models are special cases of the models for independent joint lives and so should be easy to master once you master independent joint lives.

This document briefly describes a simple way to understand (common) shock models. Not a traditional exam-prep study manual, it concentrates on explaining key ideas so that you can then understand the details presented in the textbooks or study manuals. In order to conserve space, rather than containing problems it instead lists problems for practice that can be downloaded from the SoA website starting (as of this date) at
http://www.soa.org/education/resources/edu-multiple-choice-essay-examinations.aspx .

One piece of notation I should mention. This note is full of Examples, and each example ends with the symbol ¶ at the left in a line by itself.

Chapter 1

Review: multiple lives, especially independent lives

Based on released SoA MLC/M/3 exams, it appears that a question involving shock models appears only every third exam or so, while questions on independent joint lives appear five or six times as often. Candidates sometimes decide, therefore, to skip shock models—especially since their descriptions in some text materials seem very complicated—but to master joint statuses with independent lives. But in fact the shock models are special cases of the models for independent joint lives and so should be easy to master once you master independent joint lives.

I assume that you've already studied and are in good shape on the *concepts* of the joint status on two or three lives and of the last survivor status on two or three lives, the relations among them, and especially on how you compute with those when the lives are independent—that is, when the future lifetime random variables of all the lives form an independent set. If not, go study that in your preferred textbook or study manual and then come back to this note. I'll wait. ....
....

Ready now? OK, let's begin.

1.1 Multiple lives

Even though you're familiar with joint and last-survivor statuses, I'm going to review quickly the facts I'll use the most.

The *joint status* xy (or $x : y$) on a pair of lives fails at the moment the first of x and y fails (dies), and so is *intact* while both x and y are alive. The *last-survivor status* $\overline{xy}$ (or $\overline{x:y}$) fails at the moment the last of x and y fails, and so is intact while at least one of x and y is alive. From the fact that the pair of times $\{T(x), T(y)\}$ when x and y die, respectively, must be the same as the pair of times $\{T(xy), T(\overline{xy})\}$ when the joint and last survivor statuses fail, it can be shown that almost anything "Thing" that you compute for a status—more precisely, anything computed in a ***linear*** manner from the density function f_T of the future-lifetime random variable T for that status—behaves as follows:

$$\text{Thing}(x) + \text{Thing}(y) = \text{Thing}(xy) + \text{Thing}(\overline{xy}). \tag{1.1}$$

For example,

$$_tp_x + {_tp_y} = {_tp_{xy}} + {_tp_{\overline{xy}}}$$

and

$$\overline{A}_x + \overline{A}_y = \overline{A}_{xy} + \overline{A}_{\overline{xy}}$$

and so on.

Similarly, the *joint status* xyz (or $x : y : z$) on a set of three of lives fails at the moment the first of x and y and z fails (dies), and so is *intact* while all three of x and y and z are alive. The *last-survivor status* $\overline{xyz}$ (or $\overline{x : y : z}$) fails at the moment the last of x and y and z fails, and so is intact while at least one of x and y and z is alive. The formula corresponding to Equation 1.1 in this case of three lives is of no importance for us.

For joint statuses, it's generally most convenient to work with "p functions" such as ${}_tp_{xy}$; if you have to find ${}_tp_{\overline{xy}}$ or ${}_tq_{\overline{xy}} = 1 - {}_tp_{\overline{xy}}$ in the context of the shock models of this note, it's usually wise to use Equation 1.1 and compute ${}_tp_x$, ${}_tp_y$, and ${}_tp_{xy}$ instead.

Problems 1.1

[See my Foreword on page 2 for the web link.]

From the SoA Exam MLC/M/3 archives: Spring 2005 #23; Spring 2001 #9.

1.2 Independent lives

Recall that, for *independent* lives—that is for lives whose future lifetime random variables form an independent set—the p function for the joint status is just the product of the individual p functions since the probability that the joint status survives is just the probability that all the individuals survive; for independent lives, this is just the product of the individual survival probabilities. In symbols, for two or three lives:

(1.2) for *independent* lives, $${}_tp_{xy} = {}_tp_x\, {}_tp_y, \qquad {}_tp_{xyz} = {}_tp_x\, {}_tp_y\, {}_tp_z.$$

Recall also that the force of failure or mortality for a status is defined as the negative of the time derivative of the p function divided by that p function, and that—for *independent* lives—this fact makes the force of failure for a joint status equal the sum of the forces of mortality for the individual lives. In symbols, for two or three lives:

(1.3) for *independent* lives, $$\mu_{xy}(t) = \mu_x(t) + \mu_y(t), \qquad \mu_{xyz}(t) = \mu_x(t) + \mu_y(t) + \mu_z(t).$$

Exams! ⇒ It's extremely common on Exam MLC to have all the individual forces of mortality be constants—the Exponential or forever-constant-force case. By Equation 1.3 that makes the force of failure for the joint status a constant; this means that you can use all your favorite formulas for a status u with a constant force μ, such as ${}_tp_u = e^{-\mu t}$, $\overline{A}_u = \frac{\mu}{\mu+\delta}$, and the like.

Problems 1.2

[See my Foreword on page 2 for the web link.]

From the SoA Exam MLC/M/3 archives: Fall 2006 #17; Fall 2005 #22; Fall 2004 #9; Fall 2003 #39; Fall 2002 #24; Fall 2001 #33; Fall 2000 #30.

Chapter 2

A single life subject to a shock

Now that I've reviewed the minimum basics on multiple lives and joint or last-survivor statuses, it's time to start addressing shock models.

2.1 The single-life status

First, an illustration. Suppose that you are modeling the amount $T(x)$ of future time that employee x continues to work for Daniel Enterprises. You build a model accounting for what seemed to be all reasons x might no longer be employed by Daniel: retirement, death, permanent disability, involuntary termination, and voluntary termination. And you develop your p function giving, say, $\Pr[T(x) > t] = e^{-0.1t}$. But then your friend Bubba points out that you forgot something: x might no longer be employed by Daniel Enterprises because Daniel goes bust and ceases to operate. Such an event at time t is an *external shock*—presumably no fault of x's—causing $T(x)$ to equal t, as opposed to an *internal* event such as one of the causes you originally considered.

I like to use the symbol x^* to denote the status x in the absence of the shock; of course, x truly ***is*** subject to that external force, so x^* is an idealized status that can't really exist. The p function $e^{-0.1t}$ you found is actually the p function ${}_tp_{x^*} = \Pr[T(x^*) > t]$ for this idealized status. While that's a good first step, you've still not found a model for the true $T(x)$.

NOTE: What I've called $T(x^*)$ is called $T^*(x)$ or T_x^* in most textbooks and study manuals. You'll see shortly why I like to work with the idealized status x^* instead.

Now, back to the problem of modeling the real status x rather than the idealized status x^*. What has to happen in order that x survives beyond time t? Well, x has to survive the internal causes encompassed in your model for x^*, and the external shock must not have happened. And vice versa: if x^* has survived and the external shock has not happened, then x has survived. In other words, x's survival is equivalent to that of the joint status on x^* and the external shock. If I denote the status "the external shock has not happened" by the letter z, then I've shown that $x = x^*z$. In my example of departure from Daniel Enterprises, it seems reasonable to assume that Daniel's going out of business is independent of the internal events that might cause x to depart. In short, it seems reasonable to assume that $x*$ and z are independent. This provides a definition in general.

Definition 2.1 ***(Single-life shock model)*** *A (single-life) shock model for x consists of a joint status $x = x^*z$, where x^* represents the idealized status in the absence of the possibility of an external shock, and a shock status z that represents the status "the external shock has not yet happened", with x^* and z independent.* ⇐ **KEY**

In shock-model computations, I recommend that any time you encounter the single-life status x you immediately replace it by the equivalent joint-life status x^*z, which puts you in the simple case of a joint-life status on independent lives as reviewed in Section 1.2.

While it isn't necessary to do so, the textbooks assume that the external shock status z has a

Exams! ⇒ constant force of failure denoted by λ. In symbols, $\mu_z(t) = \lambda$. Exam questions sometimes explicitly state that the force of failure for the shock status z is a constant, and sometimes they just tell you the value of λ and assume you understand what is meant. I can't recall any Exam MLC/M/3 shock-model questions that *solely* involve single lives; instead, single-life computations arise as part of the solution of problems involving a shock model for multiple lives as in this note's Chapter 3. This makes it impossible for me to suggest any single-life shock-model problems for you to solve, so I'll just give a couple of examples about how such situations might be described on an exam.

Example 2.2 An exam question might say:

> "In the absence of the shock, x would have a constant force of mortality of 0.05. However, x is part of a shock model, and the force of failure for the shock is $\mu_z(t) = 0.02$. Find ... blah blah blah ... ."

In the absence of the shock, x is the idealized status x^*, so it is x^* that has the constant force of 0.05: $\mu_{x^*}(t) = 0.05$. By Equation 1.3, this makes the force for the joint status $x = x^*z$ become $\mu_x(t) = \mu_{x^*z}(t) = \mu_{x^*}(t) + \mu_z(t) = 0.05 + 0.02 = 0.07$. The status x has a forever constant force, my favorite exam assumption.

Example 2.3 An exam question might say:

> "In the absence of the shock, a 40-year-old would have a force of mortality given by $\frac{1}{60-t}$ for $0 < t < 60$. However, 40 is part of a shock model, with $\lambda = 0.03$. Find ... blah blah blah ... ."

In the absence of the shock, 40 is the idealized status 40^*, so it is 40^* that has the force stated: $\mu_{40^*}(t) = \frac{1}{60-t}$. By Equation 1.3, this makes the force for the joint status $40 = 40^*z$ become $\mu_{40}(t) = \mu_{40^*z}(t) = \mu_{40^*}(t) + \mu_z(t) = \mu_{40^*}(t) + \lambda = \frac{1}{60-t} + 0.03$.

Problems 2.1

As I said above: I can't recall any Exam MLC/M/3 shock-model questions that *solely* involve single lives; instead, single-life computations arise as part of the solution of problems involving a shock model for multiple lives as in this note's Chapter 3. This makes it impossible for me to suggest any single-life shock-model problems for you to solve; you'll solve some as part of the solution of problems in Chapter 3.

2.2 Calculations for a single-life shock-model status

Once you have the force of mortality

KEY ⇒ (2.4)
$$\mu_x = \mu_{x^*z} = \mu_{x^*} + \mu_z$$

for a single life in a shock model, you have the complete probability model for the future lifetime $T(x)$ and can compute anything. If all you need is ${}_tp_x$, don't forget Equation 1.2: since x^* and z are independent,

KEY ⇒ (2.5)
$${}_tp_x = {}_tp_{x^*z} = {}_tp_{x^*}\,{}_tp_z.$$

Example 2.6 As in Example 2.2, an exam question might say:

> "In the absence of a shock, x would have a constant force of mortality of 0.05. However, x is part of a shock model, and the force of failure for the shock is $\mu_z(t) = 0.02$. Find ${}_5p_x$ and $\overline{A}_x$ using force of interest $\delta = 0.1$."

As you saw in Example 2.2, these assumptions make $\mu_x(t) = \mu_{x^*z}(t) = \mu_{x^*}(t)+\mu_z(t) = 0.05+0.02 = 0.07$, a forever constant force of 0.07. I can use all my favorite formulas for a status u with a constant force μ, such as ${}_tp_u = e^{-\mu t}$, $\overline{A}_u = \frac{\mu}{\mu+\delta}$, and the like. So here I get ${}_5p_x = e^{-0.07\times 5} = 0.70469$ and $\overline{A}_x = \frac{0.07}{0.07+0.1} = 0.41176$. If all I had needed was ${}_5p_x$, I could have instead computed it using Key Equation 2.5: ${}_5p_x = {}_5p_{x^*z} = {}_5p_{x^*}\,{}_5p_z = e^{-0.05\times 5}e^{-0.02\times 5} = e^{-0.07\times 5} = 0.70469$ again.

Example 2.7 As in Example 2.3, an exam question might say:

> "In the absence of a shock, a 40-year-old would have a force of mortality given by $\frac{1}{60-t}$ for $0 < t < 60$. However, 40 is part of a shock model, with $\lambda = 0.03$. Find ${}_5p_{40}$."

Since all I need is ${}_5p_{40}$, it's simplest to use Key Equation 2.5 and write ${}_5p_{40} = {}_5p_{40^*}\,{}_5p_z$. The force $\mu_{40^*}(t) = \frac{1}{60-t}$ describes a DeMoivre Law for 40^* with 60 years to go, so ${}_5p_{40^*} = 1 - \frac{5}{60} = 0.91667$. Since ${}_tp_z = e^{-\lambda t}$, I get ${}_5p_z = e^{-0.03\times 5} = e^{-0.15} = 0.86071$. Then ${}_5p_{40} = {}_5p_{40^*}\,{}_5p_z = (0.91667)(0.86071) = 0.78899$.

Problems 2.2

As I said above: I can't recall any Exam MLC/M/3 shock-model questions that *solely* involve single lives; instead, single-life computations arise as part of the solution of problems involving a shock model for multiple lives as in this note's Chapter 3. This makes it impossible for me to suggest any single-life shock-model problems for you to solve; you'll solve some as part of the solution of problems in Chapter 3.

Chapter 3

Multiple lives subject to a common shock

Look back at my motivating example about Daniel Enterprises at the start of Section 2.1. Suppose now that you have two employees x and y and that you model the times each work there in the future, ignoring the possibility of Daniel Enterprises' going out of business. This means that you model the future lifetimes of x^* and y^*, the idealized lives in the absence of the external shock z. From Chapter 2 you know that x is actually the joint status x^*z and y is actually the joint status y^*z. How can you describe the joint status xy and the last-survivor status $\overline{xy}$, and how can you compute quantities for them? You'll soon see how simple this is.

3.1 The joint and last-survivor statuses

When both x and y are subject to a common external shock, we have what is called a *common shock model.* In any such model, the textbooks and exam questions always assume that the idealized lives x^* and y^* in the absence of the possibility of the shock are *independent.* More formally:

KEY ⇒ **Definition 3.1** ***(Common shock model)*** *A common shock model for x and y consists of a joint status $x = x^*z$, where x^* represents the idealized status for x in the absence of the possibility of an external shock, a joint status $y = y^*z$, where y^* represents the idealized status for y in the absence of the possibility of an external shock, and a shock status z that represents the status "the external shock has not yet happened", with x^* and y^* and z independent.*

Since x alone constitutes a single-life shock model, you know from Chapter 2 how to compute probabilities, et cetera, for x. Likewise for y. But how about for the joint status xy and the last-survivor status $\overline{xy}$? For xy, it's easy.

What does it mean to say that the joint status xy is still intact? Precisely that x is alive and y is alive. But x is x^*z and y is y^*z; for x^*z to be surviving means that both x^* and z must be surviving, and likewise for both y^* and z. In other words, xy being intact is precisely the same thing as having all three of x^*, y^*, and z surviving. [For example, x and y both to still be working at Daniel Enterprises is equivalent to neither having left for "internal" causes and the company not having gone out of business.] In symbols: the joint status xy is identical with the joint status x^*y^*z. This deserves emphasis.

KEY ⇒ **Fact 3.2** *In the common shock model of Definition 3.1, the joint status xy is the same as the joint status x^*y^*z, with x^*, y^*, and z independent.*

In common shock computations, I recommend that any time you encounter the joint status xy you immediately replace it by the equivalent joint status x^*y^*z, which puts you in the simple case of a joint-life status on independent lives as reviewed in Section 1.2.

Because of Equation 1.3 and the independence of x^*, y^*, and z , it's easy to compute the force μ_{xy} for the joint status:

(3.3) $$\mu_{xy} = \mu_{x^*y^*z} = \mu_{x^*} + \mu_{y^*} + \mu_z = \mu_{x^*} + \mu_{y^*} + \lambda.$$ $\Leftarrow$ **KEY**

Note that in the common case on exams in which both μ_{x^*} and μ_{y^*} are constants, Key Equation 3.3 shows that μ_{xy} is also constant—my favorite forever-constant-force model again. $\Leftarrow$ **Exams!**

The preceding paragraphs show how easy it is to compute with the joint status xy in the case of common shock. How about for the last survivor status $\overline{xy}$. Your first thought might be that, since $xy = x^*y^*z$, surely it must be true that $\overline{xy}$ must be the same as $\overline{x^*y^*z}$. ***Absolutely not!***

And why not? For $\overline{xy}$ to have failed, you need both x and y to have failed, which could possibly be because both x^* and $y*$ have failed. [For example, x and y might both have retired from Daniel Enterprises.] But for $\overline{x^*y^*z}$ to have failed requires all three of x^*, y^*, and z to have failed. [For example, you'd need both x and y to have left Daniel Enterprises *and the company to have gone out of business.*] I repeat,

$\overline{xy}$ is ***not*** the same as $\overline{x^*y^*z}$. $\Leftarrow$ **KEY**

So how *can* you compute with $\overline{xy}$? By using Equation 1.1 as in the two equations immediately following that one. You'll see this in the next section.

3.2 Calculations for a common shock model

I'll start where I ended the preceding section—basic computations for the last-survivor status $\overline{xy}$. The calculation of the p function is typical of the calculation of other quantities such as single benefit premiums; the key is to calculate in terms of x alone, y alone, and the joint status xy via Equation 1.1 (as in the two equations immediately following that one):

(3.4) $${}_tp_{\overline{xy}} = {}_tp_x + {}_tp_y - {}_tp_{xy} = {}_tp_{x^*z} + {}_tp_{y^*z} - {}_tp_{x^*y^*z} = {}_tp_{x^*}\,{}_tp_z + {}_tp_{y^*}\,{}_tp_z - {}_tp_{x^*}\,{}_tp_{y^*}\,{}_tp_z.$$ $\Leftarrow$ **KEY**

And note that it's the *idea* in Key Equation 3.4, the *process,* that truly is "key". Don't just memorize the formula!

The following examples will illustrate the use of this and our other computational formulas—as well as when *not* to use the decomposition idea in Equation 1.1.

Throughout the remainder of this section, all of my examples will use the following setting—so that I don't have to keep repeating things. The phrasing is similar to that of exam questions.

In the absence of a common shock, x and y would have constant forces of mortality of 0.01 and 0.07, respectively. But they are in fact part of a common shock model with $\lambda = 0.05$. The force of interest is $\delta = 0.12$. [In my notation, this means that $\mu_{x^*}(t) = 0.01$, $\mu_{y^*}(t) = 0.07$, and $\mu_z(t) = \lambda = 0.05$.]

Example 3.5 Problem: find ${}_2p_{xy}$ and ${}_2p_{\overline{xy}}$ (using, of course, the assumptions above that hold for the remainder of this section, a fact of which I'll not remind you again!).

Solution: For the first probability, ${}_2p_{xy} = {}_2p_{x^*y^*z} = {}_2p_{x^*}\,{}_2p_{y^*}\,{}_2p_z = e^{-0.02}e^{-0.14}e^{-0.10} = e^{-0.26} = 0.77105$.

For the second probability, I use Key Equation 3.4, obtaining ${}_2p_{\overline{xy}} = {}_2p_x + {}_2p_y - {}_2p_{xy} = {}_2p_{x^*z} + {}_2p_{y^*z} - 0.77105 = {}_2p_{x^*}\,{}_2p_z + {}_2p_{y^*}\,{}_2p_z - 0.77105 = e^{-0.02}e^{-0.1} + e^{-0.14}e^{-0.1} - 0.77105 = 0.90250$. Alternatively, I could have noted that $\mu_x = \mu_{x^*z} = \mu_{x^*} + \mu_z = 0.01 + 0.05 = 0.06$, and similarly that $\mu_y = 0.07 + 0.05 = 0.12$; then ${}_2p_x = e^{-\mu_x \times 2} = e^{-0.12}$ and (similarly) ${}_2p_y = e^{-0.24}$ produce ${}_2p_{\overline{xy}} = e^{-0.12} + e^{-0.24} - 0.77105 = 0.90250$.

In the remaining examples, I'll move more quickly through the computations.

Example 3.6 Problem: find $\overline{A}_{xy}$ and $\overline{A}_{\overline{xy}}$.

Solution: For the first single benefit premium, note that μ_{xy} is constant: $\mu_{xy} = \mu_{x^*y^*z} = \mu_{x^*} + \mu_{y^*} + \lambda = 0.13$—which means that we can use the $\frac{\mu}{\mu+\delta}$ formula for a whole-life $\overline{A}$. This gives $\overline{A}_{xy} = \frac{\mu_{xy}}{\mu_{xy}+\delta} = 0.52$.

For the second single benefit premium, I rely on the idea in Equation 1.1 to write $\overline{A}_{\overline{xy}} = \overline{A}_x + \overline{A}_y - \overline{A}_{xy}$. I've already found $\overline{A}_{xy} = 0.52$. I noted in Example 3.5 that μ_x and μ_y are constants, with $\mu_x = 0.06$ and $\mu_y = 0.12$. The $\frac{\mu}{\mu+\delta}$ formula gives me $\overline{A}_x = 0.33333$ and $\overline{A}_y = 0.5$. Combining these gives $\overline{A}_{\overline{xy}} = 0.33333 + 0.5 - 0.52 = 0.31333$.

Example 3.7 Problem: find $\overline{a}_{xy}$ and $\overline{a}_{\overline{xy}}$.

Solution: One approach is to use basic principles, while another is to use the relation between $\overline{a}$ and $\overline{A}$. I'll demonstrate both.

For $\overline{a}_{xy}$, recall from the preceding Example 3.6 that μ_{xy} is a constant, with $\mu_{xy} = 0.13$. We can then use the constant-force formula $\frac{1}{\mu+\delta}$ for $\overline{a}$, getting $\overline{a}_{xy} = \frac{1}{0.13+0.12} = 4$.

I could get $\overline{a}_{\overline{xy}}$ from $\overline{a}_{\overline{xy}} = \overline{a}_x + \overline{a}_y - \overline{a}_{xy}$, but since I already know $\overline{A}_{\overline{xy}} = 0.31333$ from the preceding Example 3.6, I'll instead write $\overline{a}_{\overline{xy}} = \frac{1-\overline{A}_{\overline{xy}}}{\delta} = 5.7223$.

Example 3.8 Problem: an insurance pays \$1 at the moment of failure of the joint status xy, and premiums are paid continuously so long as both x and y are alive; find the annual benefit premium rate.

Solution: this just asks for $\overline{P}(\overline{A}_{xy}) = \frac{\overline{A}_{xy}}{\overline{a}_{xy}}$. I found $\overline{A}_{xy} = 0.52$ and $\overline{a}_{xy} = 4$ in the preceding two Examples 3.6 and 3.7, so the answer is $\overline{P}(\overline{A}_{xy}) = \frac{0.52}{4} = 0.13$. Note that this just equals μ_{xy}—for a status u with constant force μ, you always get $\overline{P}(\overline{A}_u) = \mu$.

Example 3.9 Problem: an insurance pays \$1 at the moment of failure of the last-survivor status $\overline{xy}$, and premiums are paid continuously so long as at least one of x or y is alive; find the annual benefit premium rate.

Solution: watch out! Don't try to make use of the idea in Equation 1.1 and try computing the premium rate as $\overline{P}(\overline{A}_x) + \overline{P}(\overline{A}_y) - \overline{P}(\overline{A}_{xy}) = \mu_x + \mu_y - \mu_{xy}$ (by the note at the end of Example 3.8) $= 0.06 + 0.12 - 0.13 = 0.05$. Why not? Because Equation 1.1 only holds for "Things" that depend linearly on the density functions of the future-lifetime random variables, and a continuous premium rate is the quotient $\frac{\overline{A}}{\overline{a}}$ and so is *not* linear in the densities since both the numerator and the denominator depend linearly on the density.

The correct approach is simply to compute this premium rate $\overline{P}(\overline{A}_{\overline{xy}})$ as $\frac{\overline{A}_{\overline{xy}}}{\overline{a}_{\overline{xy}}} = \frac{0.31333}{5.7223} = 0.054756$, where I've used values computed in Examples 3.6 and 3.7.

Depending on the material from which you have studied about shock models, it may come before or after the not-so-simple material on so-called simple contingent probabilities—such as the probability ${}_2q^{\ 1}_{x:y}$ that x dies before or at the same time as y and does so within the next two years. The remainder of this note and my final example consider the computation of such probabilities in the case of shock models; if you've not yet studied such simple contingent probabilities, come back to this material after you've done so.

Recall from the beginning of your MLC studies that you compute a simple probability like ${}_2q_x$ in general by an integral:

$${}_2q_x = \int_0^2 {}_tp_x\, \mu_x(t)\, dt.$$

An intuitive interpretation of the above formula is that you are adding up, over all times t between 0 and 2 when your event (the death of x) can take place, the probability the event occurs at about time t. The integral from 0 to 2 is the adding up; ${}_tp_x$ is the probability x survives out to time t so as to be able to die in the next instant; and $\mu_x(t)\,dt$ represents (intuitively) the probability x dies in the instant dt between time t and time $t+dt$, given survival to time t.

You use the same intuitive approach to compute a probability like ${}_2q_{\overset{1}{x}:y}$. To do so, for one thing you need the probability of first making it out to time t so that x can in the next instant die before or at the same time as y; this probability of making it out to time t is just ${}_tp_{xy}$. The other thing you need is the probability x dies in the next instant before or at the same time as y.

In the common shock model, you know how to compute the first needed probability: ${}_tp_{xy} = {}_tp_{x^*y^*z} = {}_tp_{x^*}\,{}_tp_{y^*}\,{}_tp_z$. With independent continuous-type future lifetime random variables, the only way to have a positive probability for x and y to die at the same time in the next instant is to have the external shock occur in the next instant; this probability, intuitively, is $\mu_z(t)\,dt = \lambda\,dt$. For x to die *strictly* first in the next instant similarly requires the idealized status x^* to fail, and this probability intuitively is $\mu_{x^*}(t)\,dt$. Adding these two terms together gives the probability x dies in the next instant before or at the same time as y, namely $\mu_z(t)\,dt + \mu_{x^*}(t)\,dt = (\mu_z(t) + \mu_{x^*}(t))\,dt = \mu_x(t)\,dt$. This gives a formula for computing ${}_2q_{\overset{1}{x}:y}$:

$$
{}_2q_{\overset{1}{x}:y} = \int_0^2 {}_tp_{xy}\,\mu_x(t)\,dt = \int_0^2 {}_tp_{x^*}\,{}_tp_{y^*}\,{}_tp_z(\mu_{x^*}(t) + \lambda)\,dt. \tag{3.10}
$$

Example 3.11 Problem: find ${}_2q_{\overset{1}{x}:y}$ using the standard assumptions of this section.

Solution: I found that $\mu_x(t) = 0.06$ in Example 3.4 and that $\mu_{xy} = 0.13$ in Example 3.5. From Equation 3.10, the solution is

$$
{}_2q_{\overset{1}{x}:y} = \int_0^2 e^{-0.13t}\,0.06\,dt = 0.10567.
$$

By the way, remember that in a common shock model the lives x and y turn out ***not*** to be independent. So you can***not*** use relations such as ${}_2q_{\overset{1}{x}:y} + {}_2q_{\overset{2}{x}:y} = {}_2q_x$ that hold for ***in***dependent lives. You ought to be able to reason intuitively in the manner above and deduce that ${}_2q_{\overset{2}{x}:y} = \int_0^2 {}_tp_x\,{}_tq_{y^*}\,\mu_x(t)\,dt$.

Problems 3.2

[See my Foreword on page 2 for the web link.]

From the SoA Exam MLC/M/3 archives: Fall 2004 #20; Spring 2001 #35; Spring 2000 #34.