Making sense of . . .

LogNormal stock-price models in Exams MFE/3 and C/4

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Foreword

This document briefly describes the ideas behind the use of LogNormal models for stock prices in some of the material for Exams MFE and C of the Society of Actuaries and Exams 3 and 4 of the Casualty Actuarial Society. Not a traditional exam-prep study manual, it concentrates on explaining key ideas so that you can then understand the details presented in the textbooks or study manuals. It can be especially useful to anyone taking Exam C/4 without having studied the material for Exam MFE/3.

Chapter 1

LogNormal stock-price models

1.1 Why LogNormal models?

Why learn about and use LogNormal models for stock prices? I could answer "Because it's on the exam syllabi" or "Why not?", but that wouldn't be helpful. Instead, I'll take a little space to motivate this.

Suppose that the price of a stock or other asset at time 0 is known to be $S(0)$ and we want to model its future price $S(10)$ at time 10—note that some texts use the notation S_0 and S_{10} instead. Let's break the time interval from 0 to 10 into 10,000 pieces of length 0.001, and let's let S_k stand for $S(0.001k)$, the price at time $0.001k$. I know the price $S_0 = S(0)$ and want to model the price $S_{10000} = S(10)$. I can write

$$S(10) = S_{10000} = \frac{S_{10000}}{S_{9999}} \frac{S_{9999}}{S_{9998}} \cdots \frac{S_2}{S_1} \frac{S_1}{S_0} S_0. \tag{1.1}$$

Now suppose that the ratios $R_k = \frac{S_k}{S_{k-1}}$ that appear in Equation 1.1 that represent the growth factors in price over each interval of length 0.001 are random variables, and—to get a simple model—are all independent of one another.

Then Equation 1.1 writes $S(10)$ as a *product* of a large number of independent random variables R_k. You know from probability that the *sum* of a large number of random variables W_k can, under reasonable hypotheses, be approximated well by a Normal random variable with the same mean and variance as the sum. Unfortunately Equation 1.1 involves a product, not a sum. But if we take the natural log of both sides, we get

$$\ln S_{10000} = \ln R_{10000} + \ln R_{9999} + \cdots + \ln R_2 + \ln R_1 + \ln S_0. \tag{1.2}$$

This equation represents $\ln S(10)$ as $\ln S_0 = \ln S(0)$ plus the sum of a large number of independent random variables. Under reasonable hypotheses such as that all the $\ln R_k$ have the same probability distribution with positive variance, this implies that $\ln S(10)$ is well approximated by $\ln S(0)$ plus a Normal random variable with mean and variance each 10000 times that of a typical $\ln R_k$.

Note that if I took this same approach on the interval from time 0 to time 20, the mean and variance of the Normal would be 20000 times that of $\ln R_k$—when I doubled the length of the time interval, I doubled the mean and variance of the Normal. That is, the mean and variance of the Normal grow linearly with the length of the time interval.

So it seems that a plausible model for the price $S(t)$ of a stock at time t is that $\ln S(t)$ should be $\ln S(0)$ plus a Normal random variable with mean and variance both proportional to t:

$$\ln S(t) = \ln S(0) + \mathcal{N}(\mu t, \sigma^2 t), \tag{1.3}$$

where $\mathcal{N}(m, s^2)$ denotes a Normal random variable with mean m and variance s^2. Exponentiating both sides of Equation 1.3 gives

$$S(t) = S(0)e^{\mathcal{N}(\mu t, \sigma^2 t)}, \tag{1.4}$$

which models $S(t)$ as S(0) times a LogNormal random variable with parameters μt and $\sigma^2 t$. This indicates that LogNormal models (with linearly growing parameters) are plausible models for the price of stocks and other assets.

More precisely, the LogNormal models are examples of so-called ***Geometric Brownian Motion;*** the remaining sections of this note explain in more detail the precise nature of these models and how to calculate with them.

1.2 The first step—Standard Brownian motion B

Recall that for SoA Exam MLC or CAS Exam 3 you learned about Poisson Processes, a particular type of stochastic process—that is, a collection of random variables—that had independent increments. You don't need to remember about Poisson Processes, just stochastic processes with independent increments. To be cautious, I'll review that for you.

A *stochastic process* X for $t \geq 0$ is such that, for each value of $t \geq 0$, $X(t)$ is a random variable. For example, $X(t)$ might represent the number of insurance claims filed with a particular insurance company by time t, or the value of all those claims, or the interest rate at time t, or, well, whatever entity you might want to model.

To understand the ***Standard Brownian Motion*** stochastic process B, you need to understand the meaning and probability behavior of the *increment* $B(t+h) - B(t)$ from time t to time $t+h$, where $h > 0$ and of course $t \geq 0$. The increment is simply the change in the value of B from time t to time $t+h$. Note that—since you'll soon see that $B(0) = 0$—the value $B(t) = B(t) - B(0)$ itself can be viewed as an increment, namely from time 0 to time t.

A fundamental property of Standard Brownian Motion processes is that increments on non-overlapping time intervals are independent of one another as random variables—stated intuitively, knowing something about the change in B over one interval gives you no information about the change over a non-overlapping interval. But notice the important modifier "non-overlapping". While the increments $B(5.2) - B(3.1)$ and $B(2.7) - B(1.6)$ ***are*** independent, $B(5.2) - B(3.1)$ and $B(4.3) - B(3.7)$ are ***not*** independent since the time intervals overlap. Similarly, the increments $B(5.2) - B(3.1)$ and $B(3.8) - B(1.5)$ are ***not*** independent since the time intervals overlap. Note, however, that $B(5.2) - B(3.1)$ and $B(3.1) - B(2.4)$ ***are*** independent since the intervals do not overlap; that is, touching at an endpoint is "OK"—time intervals that touch *only* at an endpoint *do* have independent increments.

KEY ⇒ **Definition 1.5 (Standard Brownian Motion)** *Standard Brownian Motion B has the following properties:*

1. *B is a stochastic process with $B(0) = 0$;*

2. *B has independent increments—any set of increments $B(t_j + h_j) - B(t_j)$ for $j = 1, 2, \ldots, n$ is independent, provided that the time intervals $(t_j, t_j + h_j]$ are non-overlapping (touching only at an endpoint is OK);*

3. *for all $t \geq 0$ and $h > 0$, the increment $B(t+h) - B(t)$ is a Normal random variable with mean 0 and variance h.*

Let's look at an example of how the properties of Standard Brownian Motion are used, especially that of independent increments.

Example 1.6 Suppose that B is a Standard Brownian Motion process. Also suppose that you've observed $B(1.21) = 3.4$. [By the way, this value is surprisingly far from its mean, since $B(1.21) = B(1.21) - B(0)$ is a Normal random variable with mean 0 and variance 1.21, so standard deviation 1.1, revealing that $B(1.21)$ is over three standard deviations away from its mean.] Taking this observation of $B(1.21)$ into account, you want to understand the probability behavior of $B(1.85)$. You might (momentarily) wonder whether you should expect B to continue to be surprisingly far from its mean. But then you remember that Standard Brownian Motion has independent increments, so knowing what happened from time 0 to time 1.21 should have no impact on what happens from time 1.21 to time 1.85.

More precisely, write $B(1.85)$ as

$$B(1.85) = [B(1.85) - B(1.21)] + [B(1.21) - B(0)].$$

Given the observation that $B(1.21) = 3.4$, you know that the second term in square brackets [ ] equals 3.4. Because of independent increments, the first term in square brackets (given the observation) is simply a Normal random variable—call it W—with mean 0 and variance $1.85 - 1.21 = 0.64$. Thus, given the observation, the distribution of $B(1.85)$ is given by

$$[B(1.85) \,|\, B(1.21) = 3.4] \sim W + 3.4,$$

with $W \sim \mathcal{N}(0, 0.64)$ a Normal random variable with mean 0 and variance 0.64 (and with $\sim$ meaning "has the same probability distribution as").

This makes it easy to translate probability questions about $B(1.85)$, given $B(1.21) = 3.4$, to questions about W. For example, for the mean

$$\mathrm{E}[B(1.85) \,|\, B(1.21) = 3.4] = \mathrm{E}[W + 3.4] = \mathrm{E}[W] + 3.4 = 0 + 3.4 = 3.4,$$

and for the variance

$$\mathrm{Var}[B(1.85) \,|\, B(1.21) = 3.4] = \mathrm{Var}[W + 3.4] = \mathrm{Var}[W] = 0.64;$$

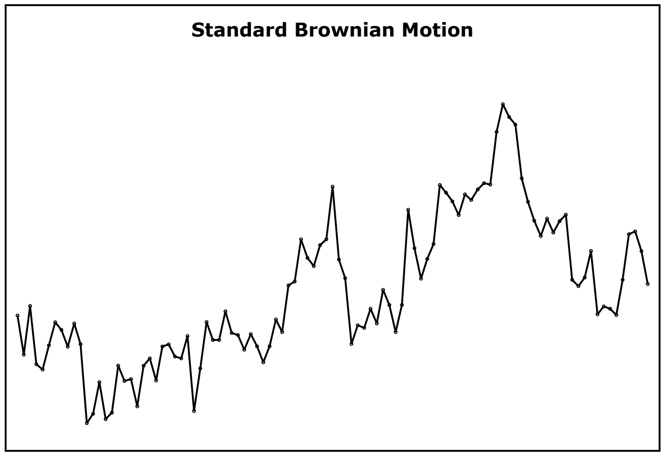
the equality of $\mathrm{Var}[W + 3.4]$ and $\mathrm{Var}[W]$ in the above followed from the fact that adding a constant to a random variable shifts all its random values by the same amount but not how those values vary among themselves. You compute probabilities just as easily; for example,

$$\begin{aligned} \Pr[B(1.85) \le 2.6 \,|\, B(1.21) = 3.4] &= \Pr[W + 3.4 \le 2.6] = \Pr[W \le -0.8] = \Pr[\mathcal{N}(0, 0.64) \le -0.8] \\ &= \Pr[\mathcal{N}(0, 0.64)/0.8 \le -1] = \Pr[\mathcal{N}(0, 1) \le -1] \approx 0.1587, \end{aligned}$$

using a table of probabilities for the standard Normal distribution. ¶

On the next page you'll find the graph of a simulated approximation to a Standard Brownian Motion for $0 \le t \le 1$.

(1.7)



1.3 The second step—Arithmetic Brownian Motion A

Standard Brownian Motion always has mean 0 but has linearly growing variance equal to t. ***Arithmetic Brownian Motion***—also often called ***Brownian Motion with Drift***—allows for linearly growing mean and for the variance to be *proportional* to t rather than simply *equal* to t.

Definition 1.8 (Arithmetic Brownian Motion) *Arithmetic Brownian Motion A with drift parameter μ, volatility parameter σ, and initial value A_0 is a stochastic process A with $A(t) = A_0 + \mu t + \sigma B(t)$, where B is Standard Brownian Motion.*

It's easy to see how increments of A behave, since they are completely determined by corresponding increments of B. That is,

$$A(t+h) - A(t) = [A_0 + \mu(t+h) + \sigma B(t+h)] - [A_0 + \mu t + \sigma B(t)] = \mu h + \sigma[B(t+h) - B(t)].$$

Since $B(t+h) - B(t) \sim \mathcal{N}(0, h)$, this makes $A(t+h) - A(t) \sim \mathcal{N}(\mu h, \sigma^2 h)$. Relating increments in A to increments in B in this manner and using Key Definition 1.5 on Standard Brownian Motion leads to the following key facts—which you should compare with the correspondingly numbered facts about Standard Brownian Motion in Key Definition 1.5.

KEY ⇒ **Fact 1.9 (Arithmetic Brownian Motion properties)** *Suppose that A is an Arithmetic Brownian Motion with drift parameter μ, volatility parameter σ, and initial value A_0. Then:*

1. *A is a stochastic process with $A(0) = A_0$;*

2. *A has independent increments—any set of increments $A(t_j + h_j) - A(t_j)$ for $j = 1, 2, \ldots, n$ is independent, provided that the time intervals $(t_j, t_j + h_j]$ are non-overlapping (touching only at an endpoint is OK);*

3. *for all $t \geq 0$ and $h > 0$, the increment $A(t+h) - A(t)$ is a Normal random variable $\mathcal{N}(\mu h, \sigma^2 h)$ with mean μh and variance $\sigma^2 h$.*

Note that Standard Brownian Motion is just Arithmetic Brownian Motion with drift parameter 0, volatility parameter 1, and initial value 0.

Example 1.10 Suppose that A is an Arithmetic Brownian Motion with drift parameter $\mu = -1$, volatility parameter $\sigma = 2$, and initial value $A_0 = 3$. Suppose that you've observed that $A(1.21) = 5$ and you wish to know the behavior of the random variable $A(1.85)$—that is, of $A(1.85) \,\big|\, A(1.21) = 5$. Simply write $A(1.85) = A(1.85) - A(1.21) + A(1.21) = A(1.85) - A(1.21) + 5$ and then use Item 3 of Fact 1.9, namely that $A(1.85) - A(1.21)$ is Normal with mean $(-1)(1.85 - 1.21) = -0.64$ and variance $(2^2)(1.85 - 1.21) = 2.56$. Since $A(1.85)$ is just this Normal plus 5, producing a Normal with the same variance but the mean increased by 5, $[A(1.85) \,\big|\, A(1.21) = 5] \sim \mathcal{N}(4.36, 2.56)$, a Normal with mean 4.36 and variance 2.56. ¶

Note that, in general, by arguments like those in Example 1.10,

$$(1.11)\quad A(t) = A(t) - A(0) + A(0) = A(t) - A(0) + A_0 \sim \mathcal{N}(\mu t, \sigma^2 t) + A_0 \sim \mathcal{N}(A_0 + \mu t, \sigma^2 t).$$

1.4 The final step—Geometric Brownian Motion G

A ***Geometric Brownian Motion*** G is nothing other than what you get by exponentiating an Arithmetic Brownian Motion A—that is, $G = e^A$. You know from Equation 1.11 above that $A(t)$ is Normal with linearly growing mean $A_0 + \mu t$ and linearly growing variance $\sigma^2 t$—that is, $A(t) \sim \mathcal{N}(A_0 + \mu t, \sigma^2 t)$. Since $G(t)$ is then just the exponential of a Normal random variable, it's a LogNormal random variable.

Note also that $G_0 = G(0) = e^{A(0)} = e^{A_0}$; equivalently, $A_0 = \ln G_0$. Definition 1.8 on Arithmetic Brownian Motion defined A via $A(t) = A_0 + \mu t + \sigma B(t)$. This makes

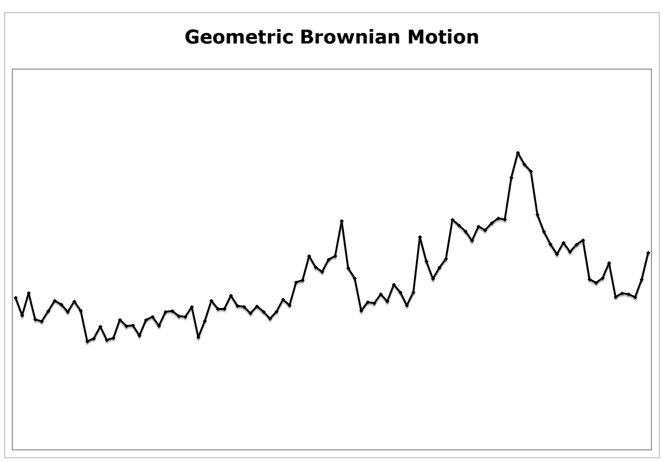
$$G(t) = e^{A(t)} = e^{A_0 + \mu t + \sigma B(t)} = e^{\ln G_0 + \mu t + \sigma B(t)} = G_0 e^{\mu t + \sigma B(t)}.$$

This last is the form I'll use in the definition.

Definition 1.12 (Geometric Brownian Motion) *Geometric Brownian Motion G with drift parameter μ, volatility parameter σ, and initial value G_0 is a stochastic process G with $G(t) = G_0 e^{\mu t + \sigma B(t)}$, where B is Standard Brownian Motion.* ⇐ **KEY**

Here's the graph of a simulated approximation to a Geometric Brownian Motion for $0 \leq t \leq 1$, with $G(0) = 1$; it results from exponentiating the simulated Standard (thus Arithmetic) Brownian Motion shown in Graph 1.7.

(1.13)



Increments $A(t+h)-A(t)$ played a key role with Arithmetic (and Standard) Brownian Motion. For Geometric Brownian Motion, *growth factors* $\frac{G(t+h)}{G(t)}$ play a key role. Why? Because a growth factor $\frac{G(t+h)}{G(t)}$ for G is an exponentiated increment of an Arithmetic Brownian Motion:

$$\frac{G(t+h)}{G(t)} = \frac{G_0 e^{\mu(t+h)+\sigma B(t+h)}}{G_0 e^{\mu t+\sigma B(t)}} = e^{\mu h+\sigma[B(t+h)-B(t)]}. \tag{1.14}$$

This relationship produces the following three facts about growth factors for Geometric Brownian Motion from the corresponding facts in Key Definition 1.5 on Standard Brownian Motion or in Fact 1.9 about increments in Arithmetic Brownian Motion; the second part of the third fact follows from recalling—or looking in the tables for Exams 3 and C/4 to see—that the mean and variance of a LogNormal $W = e^{\mathcal{N}(m,s^2)}$are given by

$$E[W] = e^{m+\frac{1}{2}s^2}, \text{ and } \operatorname{Var}[W] = \{E[W]\}^2(e^{s^2}-1). \tag{1.15}$$

Fact 1.16 (Geometric Brownian Motion properties) *Suppose that G is a Geometric Brownian Motion with drift parameter μ, volatility parameter σ, and initial value G_0. Then:*

1. *G is a stochastic process with $G(0) = G_0$;*
2. *G has independent growth factors—any set of growth factors $G(t_j+h_j)/G(t_j)$ for $j = 1, 2, \dots, n$ is independent, provided that the time intervals $(t_j, t_j+h_j]$ are non-overlapping (touching only at an endpoint is OK);*
3. *for all $t \geq 0$ and $h > 0$, the growth factor $G(t+h)/G(t)$ is a LogNormal random variable $e^{\mathcal{N}(\mu h,\sigma^2 h)}$ with mean $e^{(\mu+\frac{1}{2}\sigma^2)h}$ and variance $e^{(2\mu+\sigma^2)h}(e^{\sigma^2 h}-1)$.*

Since $G(t) = \frac{G(t)}{G(0)}G_0$, item 3) in Fact 1.16 implies that

$$\mathrm{E}[G(t)] = G_0 e^{(\mu+\frac{1}{2}\sigma^2)t}. \tag{1.17}$$

When textbooks in financial economics (or financial mathematics or mathematical finance) such as the *Derivatives Markets* book for the actuarial exams MFE/3 and C/4 use a Geometric Brownian Motion $G(t)$ at time t to model a stock price $S(t)$ at time t in terms of $S(0) = S_0$, they sometimes use α to denote $\mu + \frac{1}{2}\sigma^2$. Then Equation 1.17 becomes

$$\mathrm{E}[S(t)] = S_0 e^{\alpha t}. \tag{1.18}$$

The parameter α above can then be interpreted as the continuously compounded expected rate of return on the stock. The parameter σ still represents volatility, and the drift parameter μ is calculated as $\mu = \alpha - \frac{1}{2}\sigma^2$. More generally, if the stock pays dividends and δ denotes the continuously compounded rate of dividends on the stock, you'll see in financial economics that the drift parameter μ is calculated as $\mu = \alpha - \frac{1}{2}\sigma^2 - \delta$.

1.5 Calculations on exams

Section 1.1 explained why LogNormal models are at least plausible for stock prices, and Section 1.4 finished the development of Geometric Brownian Motion as a more precise description of these LogNormal models. Throughout this final section I consider a Geometric Brownian Motion S as a model of a stock price; it has drift parameter μ, volatility σ, and initial value $S(0) = S_0$. Remember from the end of the preceding section that μ might be computed as $\mu = \alpha - \frac{1}{2}\sigma^2$ or $\mu = \alpha - \frac{1}{2}\sigma^2 - \delta$, where α is the continuously compounded expected rate of return on the stock and δ is the continuously compounded rate of dividends on the stock, as in the *Derivatives Markets* textbook for Exams MFE/3 and C/4.

Recall from Definition 1.12 on Geometric Brownian Motion that you can view S as

$$S(t) = S_0 e^{\mu t + \sigma B(t)},$$

where B is Standard Brownian Motion. Actuarial exam questions may well ask you to perform calculations involving $S(t)$;

> *in many cases you can decline and instead calculate with $B(t)$, and I strongly recommend that you do so* ⇐ **Exams!**

since $B(t)$ is just a Normal random variable. I'll show you what I mean in the following subsections.

1.5.1 Probabilities

Suppose you are asked to compute $\Pr[S(t) \le z]$, say. Instead of directly dealing with the LogNormal random variables $S(t)$, write

$$\Pr[S(t) \le z] = \Pr[S_0 e^{\mu t + \sigma B(t)} \le z] = \Pr\left[B(t) \le \frac{\ln(z/S_0) - \mu t}{\sigma}\right] \tag{1.19}$$

and deal with the Normal random variable $B(t) \sim \mathcal{N}(0, t)$.

Example 1.20 Suppose that $\mu = -1.9$, $\sigma = 2$, and $S_0 = 100$, and that you need to compute the probability $\Pr[S(2) \le 105]$. Repeating the development in Equation 1.19 with these values shows that the probability is given by

$$\begin{aligned}\Pr\left[B(2) \le \frac{\ln(105/100) - (-1.9)(2)}{2}\right] &= \Pr\left[\mathcal{N}(0,2) \le \frac{\ln(1.05) + 3.8}{2} = 1.9244\right] \\ &= \Pr\left[\mathcal{N}(0,1) \le \frac{1.9244}{\sqrt{2}} = 1.3608\right] \approx 0.9131. \quad \P\end{aligned}$$

1.5.2 Simulation

The syllabus for Exam C/4 contains material on simulation by Inversion. To Inversion simulate a continuous-type random variable W with strictly increasing cumulative distribution function, you obtain a value U from a Uniform random variable on the interval from 0 to 1, solve the equation $\Pr[W \le z] = U$ for z, and set the Inversion-simulated value of W to be z; I'll denote this Inversion-simulated value by $ISim\ W$.

Example 1.21 Suppose you need to simulate a standard Normal random variable $Z \sim \mathcal{N}(0, 1)$ by Inversion. Given a Uniform(0,1) value U, you need to find the number z so that $\Pr[\mathcal{N}(0, 1) \le z] = U$; such a number z can be found from tables for the standard Normal distribution, and let's call this value z_U. Then

$$ISim\ \mathcal{N}(0,1) = z_U, \text{where } \Pr[\mathcal{N}(0,1) \le z_U] = U.$$

For example, if $U = 0.975$, then $z_U = 1.96$. ¶

A useful fact about simulation by Inversion is that, if random variables X and Y are related by $Y = g(X)$ for some non-decreasing function g, then for a given value U of a Uniform(0,1) random variable it turns out that $ISim\ Y$ and $g(ISimX)$ are identical. This is the *monotonic transformation property* of (simulation by) Inversion.

Example 1.22 Suppose that you need to Inversion simulate a Normal random variable $W \sim \mathcal{N}(m, s^2)$ with mean m and variance s^2. We can think of W as $W = g(Z) = m + sZ$ for a standard Normal random variable $Z \sim \mathcal{N}(0, 1)$. According to the monotonic transformation property of Inversion in the preceding paragraph, $ISim\ W = m + sISim\ Z$. But by Example 1.21, $ISim\ Z = z_U$. Thus $ISim\ W = m + sz_U$. That is, $ISim\ \mathcal{N}(m, s^2) = m + sz_U$. ¶

You may be asked to Inversion simulate values of $S(t)$. Remember that $S(t) = S_0 e^{\mu t + \sigma B(t)}$. By the monotonic transformation property of Inversion you'll get precisely the same result by Inversion simulating $B(t)$ and computing $S(t)$ from it as by Inversion simulating $S(t)$ directly. And $B(t)$ is easy to Inversion simulate as in Example 1.22 since $B(t) \sim \mathcal{N}(m, s^2)$ with $m = 0$ and $s^2 = t$. You simply get a Uniform(0,1) value U and compute

$$ISim\ B(t) = 0 + \sqrt{t}\, z_U = \sqrt{t}\, z_U.$$

Thus the efficient way to Inversion simulate $S(t)$ is to Inversion simulate $B(t)$ as above and then compute

$$ISim\ S(t) = S_0 e^{\mu t + \sigma ISim\ B(t)}.$$

In practice—and on exams—you seldom want to simulate just a single value $S(t)$; rather you want to simulate a sequence of values $S(t_1)$, $S(t_2)$, $\ldots$, $S(t_n)$, with $0 < t_1 < t_2 < \cdots < t_n$. Since each $S(t_j) = S_0 e^{\mu t_j + \sigma B(t_j)}$, all you need do is simulate each $B(t_j)$ and then compute $S(t_j)$ from that value. Suppose that you simulate $B(t_1)$, possibly by Inversion. Since the intervals $(0, t_1]$ and $(0, t_2]$ overlap, $B(t_2)$ is dependent on $B(t_1)$ and this dependence must be taken into account in simulating $B(t_2)$. Recall from Key Definition 1.5 on Standard Brownian Motion that B has independent increments on non-overlapping time intervals. So $B(t_2) - B(t_1)$ can be simulated independently from $B(t_1) = B(t_1) - B(0)$, and similarly for each increment $B(t_{j+1}) - B(t_j)$. And don't forget that—letting h_j denote $t_{j+1} - t_j$—you know how to Inversion simulate $B(t_{j+1}) - B(t_j) = B(t_j + h_j) - B(t_j)$ since by Key Definition 1.5 on Standard Brownian Motion this increment is just a Normal random variable $\mathcal{N}(0, h_j)$ and you can Inversion simulate Normal random variables as in Example 1.22. This process is sometimes described as "Inversion in steps". An example should clarify how you do this.

Example 1.23 Suppose that $\mu = 0.06$, $\sigma = 2$, and $S_0 = 100$, and that you need to simulate $S(t)$ at $t_1 = 0.5$ and $t_2 = 1$ in steps of $h = 0.5$ using the two independent Uniform(0,1) values $U_1 = 0.4602$ and $U_2 = 0.6554$, respectively. I'll do so as described in the preceding paragraph by simulating $B(t)$ at $t_1 = 0.5$ and $t_2 = 1$ using Inversion on the increments in steps of $h = 0.5$ and then computing $S(t)$ as $S(t) = S_0 e^{\mu t + \sigma B(t)}$. Watch.

For convenience I'll denote the first increment $B(0.5) = B(0.5) - B(0)$ by I_1 and the second $B(1) - B(0.5)$ by I_2. Since $B(0) = 0$, you also can say that $B(0.5) = I_1$ and $B(1) = I_1 + I_2$. As explained in the preceding paragraph, both I_1 and I_2 are Normal random variables $\mathcal{N}(0, 0.5)$ with mean 0 and variance 0.5. As in Example 1.22, each is therefore Inversion simulated as $ISim\ \mathcal{N}(0, 0.5) = 0 + \sqrt{0.5}\, z_U \approx 0.70711 z_U$. For our two values of U, tables of the standard Normal distribution give $z_{0.4602} = -0.1$ and $z_{0.6554} = 0.4$. This gives $ISim\ I_1 \approx 0.70711(-0.1) = 0.070711$ and $ISimI_2 \approx 0.70711(0.4) = 0.28284$. Thus, the simulated values $\widehat{B}(0.5)$ of $B(0.5) = I_1$ and $\widehat{B}(1)$ of $B(1) = I_1 + I_2$ are $\widehat{B}(0.5) = -0.070711$ and $\widehat{B}(1) = -0.070711 + 0.28284 \approx 0.21213$, respectively. So my simulated value $\widehat{S}(0.5)$ of $S(0.5)$ is

$$\widehat{S}(0.5) = S_0 e^{\mu \times 0.5 + \sigma \widehat{B}(0.5)} = 100 e^{(0.06)(0.5) + (2)(-0.070711)} \approx 89.456.$$

Similarly, my simulated value $\widehat{S}(1)$ of $S(1)$ is

$$\widehat{S}(1) = S_0 e^{\mu \times 1 + \sigma \widehat{B}(1)} = 100 e^{(0.06)(1) + (2)(0.21213)} \approx 162.30. \quad ¶$$

1.5.3 Options and option pricing

The syllabi for Exams MFE/3 and C/4 contain material on European call options and European put options with strike price K at time T on a stock having price modeled by $S(t)$. As a quick reminder:

1. A *European call option with strike price K at time T* gives the option owner the right to **buy** one share of the stock at time T for the price K if the owner wishes to do so. If the stock price $S(T)$ then is above K, the owner will exercise the option, buy the share for K and immediately sell it for $S(T)$, making a net amount $N = S(T) - K$. If the price is at or below K, the owner won't exercise the option and makes a net amount $N = 0$. More compactly, I'll write the option owner's net amount N at time T as $N = (S(T) - K)_+$, where the expression $(Y)_+$ denotes the non-negative part of Y: $(Y)_+$ equals Y if $Y \geq 0$ and equals 0 if $Y < 0$.

2. A *European put option with strike price K at time T* gives the option owner the right to **sell** one share of the stock at time T for the price K if the owner wishes to do so. If the stock price $S(T)$ then is below K, the owner will buy the share for $S(T)$ and immediately exercise the option by selling the share for K, making a net amount $N = K - S(T)$. If the price is at or above K, the owner won't exercise the option and makes a net amount $N = 0$. More compactly, I'll write the option owner's net amount N at time T as $N = (K - S(T))_+$.

If you are asked to simulate the net amount N on an option, you merely simulate the stock price $S(T)$ at time T as discussed in the preceding subsection and then compute the net amount.

The famous Black-Scholes formulas compute the price to purchase one of the above options at time 0 as the expected value of the present value—called the actuarial present value in the life contingencies material on Exam MLC/3—of the net amount N. In the *Derivatives Markets* textbook, the force of interest used in discounting to obtain present values is denoted by r and called the risk-free rate; moreover, the continuously compounded expected rate of return α on the stock is assumed to equal the risk-free rate r, so $\alpha = r$. For consistency with that text, I now assume that the stock price S is Geometric Brownian Motion with drift parameter $\mu = r - \frac{1}{2}\sigma^2 - \delta$, volatility σ, and initial value S_0, where r is the risk-free force of interest used in discounting and the continuously compounded expected rate of return on the stock.

The Black-Scholes price C for a European call option with strike price K at time T is then computed as

$$C = \mathrm{E}[e^{-rT}N] = e^{-rT}\,\mathrm{E}[(S(T) - K)_+], \tag{1.24}$$

while the price P for a European put option with strike price K at time T is computed as

$$P = \mathrm{E}[e^{-rT}N] = e^{-rT}\,\mathrm{E}[(K - S(T))_+]. \tag{1.25}$$

So the question is: on exams, how do you compute C or P above?

> For exam MFE, unfortunately you should memorize the formulas from the textbook or your study manual. But for Exams 3 and C/4, you can use information in the distribution tables provided and needn't memorize any formulas.

⇐ **Exams!**

To use the tables, you need to remember a relationship also useful in the Exam C/4 syllabus material on severity. Recall that the notation $A \wedge B$ stands for the smaller of A and B. The useful relationship is that

$$(A - B)_+ + (A \wedge B) = A \tag{1.26}$$

for all numbers A and B. [You should check this by considering the two cases $A \geq B$ and $A < B$.] To use this for the call price C in Equation 1.24, for example, write $(S(T) - K)_+ + (S(T) \wedge K) = S(T)$, from which it follows that $(S(T) - K)_+ = S(T) - (S(T) \wedge K)$. Therefore the expected value required for C in Equation 1.24 can be computed as

$$\mathrm{E}[(S(T) - K)_+] = \mathrm{E}[S(T)] - \mathrm{E}[S(T) \wedge K]. \tag{1.27}$$

So calculating the price C of the European call option reduces to calculating the two expected values on the right-hand side of Equation 1.27. How do you do that? By looking them up in the tables for Exams 3 and C/4! Remember that $S(T)$ is a LogNormal random variable, and the tables contain formulas for the moments $\mathrm{E}[W^k]$ and limited moments $\mathrm{E}[(W \wedge x)^k]$ of a LogNormal random variable W.

Example 1.28 Suppose that the initial stock price is $S_0 = 41$, the strike price on a European call option at time $T = 0.25$ is $K = 40$, the volatility is $\sigma = 0.3$, the risk-free force of interest (and continuously compounded expected rate of return on the stock) is $r = 0.08$, and that there are no dividends—that is, that $\delta = 0$. This yields a drift parameter $\mu = r - \frac{1}{2}\sigma^2 - \delta = 0.035$. I want to compute the Black-Scholes call price C in Equation 1.24 via

$$C = e^{-(0.08)(0.25)}\,\mathrm{E}[(S(0.25) - 40)_+].$$

Now

$$\begin{aligned} S(0.25) &= S_0 e^{\mu \times 0.25 + \sigma B(0.25)} = 41 e^{(0.035)(0.25) + (0.3)B(0.25)} \\ &= e^{\ln 41 + 0.00875 + 0.3\mathcal{N}(0, 0.25)} = e^{\mathcal{N}(3.7223,\, 0.0225)}, \end{aligned}$$

which shows that $S(0.25)$ is a LogNormal random variable $W = e^{\mathcal{N}(m,s^2)}$ with $m = 3.7223$ and $s^2 = 0.0225$ (so $s = 0.15$). To use Equation 1.27, I need both $\mathrm{E}[W]$ and $\mathrm{E}[W \wedge 40]$. The Exams 3 and C/4 tables for the LogNormal give

$$\mathrm{E}[W] = e^{m + \frac{1}{2}\sigma^2} \approx e^{3.73355} \approx 41.827;$$

similarly, they give

$$\mathrm{E}[W \wedge 40] = e^{m + \frac{1}{2}s^2}\,\Phi\left(\frac{\ln 40 - m - s^2}{s}\right) + 40\left[1 - \Phi\left(\frac{\ln 40 - m}{s}\right)\right],$$

where $\Phi(y) = \Pr[\mathcal{N}(0,1)] \leq y]$ gives the cumulative distribution function for the standard Normal distribution. Calculating produces $\mathrm{E}[W \wedge 40] \approx 38.364$. So, from Equation 1.27,

$$\mathrm{E}[S(0.25) - 40)_+] = \mathrm{E}[(W - 40)_+] = \mathrm{E}[W] - \mathrm{E}[W \wedge 40] \approx 41.827 - 38.364 = 3.463.$$

Finally, from Equation 1.28 we get the price of the European call as

$$C \approx e^{-0.02}(3.463) \approx 3.395.$$

Tedious, but no memorized formulas! ¶